ON THE CHARACTERIZATION AND REDUCTION OF ILL-CONDITIONING IN JOINTLY ESTIMATING PROTON DENSITY, T_2^* DECAY AND THE FIELD MAP

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***Abstract*—We employ principal angles as a tool to demonstrate that the complex frequency is the main cause for ill-conditioning of model-based MRI reconstruction. We show that kernels of EPI and spiral trajectories have different null sets that can be utilized to improve the conditioning. This idea is further verified based on synthesized phantom data.**

I. INTRODUCTION

In many applications we desire to estimate T_2^* decay, off-resonance frequency field map and proton density map through a few scans. However, the joint reconstruction of all these parameters can be severely ill conditioned [1]. Regularization can reduce the issue but it involves a tradeoff between conditioning and resolution. Adding a trajectory adds minimal acquisition time and does not sacrifice the resolution of the image. We show that this scheme can improve the conditioning significantly, making it an attractive strategy.

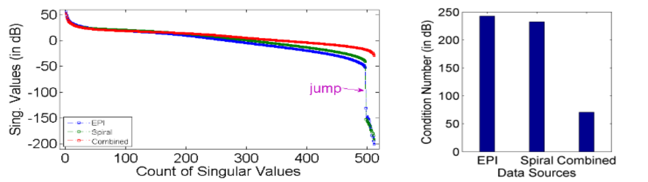
II. THEORY

The underlying signal model in this context is

$$s_l = \int_r a(r) e^{b(r)t_l} e^{-2\pi j[k_l \cdot r]} dr + \varepsilon_l \quad l = 1,2 \cdots L \quad (1)$$

where $(a(r), b(r))$ is a realization of the complex amplitude and frequency, which includes as real and imaginary parts the T_2^* decay and off-resonance frequency. s_l, ε_l, t_l *and* k_l are the k-space signal, noise, time samples and trajectory samples. For both EPI and spiral, the condition number (CN) can be more than 10^{21} and the singular value curve is discontinuous (Fig 1). According to Hansen [2], this type of mapping is characterized as rank-deficient—a numerical null set exists such that any variable offset within this subspace induces only a negligible variation of the model residual.

The cause of the high CN is critical. The conception of principal angles (PA) is a generalization of the angle between two straight lines to that between two hyperplanes. In this work, the PA is used to determine which variable set—amplitude or frequency—has a greater overlap with the previously mentioned numerical null set. The result shows that the frequency almost completely overlaps with the null set (PA=1.30˚±0.69˚) while amplitude is almost orthogonal



(a) (b)

Fig 1. Mean singular value curve(a) and mean CN(b) for three data sources.

(PA=88.70˚±0.69˚). Frequency variation therefore causes much more variation within the null set. That is, frequency is the main cause of the poor conditioning. This holds for both EPI and spirals. The size of our simulated images was 16x16, but the observations also hold for larger sizes.

Mean PA between null sets of EPI and spirals is 53.17˚ ± 7.24˚, showing that EPI and spirals act as different encodings of variables $(a(r), b(r))$. By combining the two trajectories, the reduction of CN is roughly 10^{15} (Fig 1(b)) and the jump of the singular value curve is removed (Fig 1(a)). Better conditioning generally leads to faster convergence and higher precision. This is verified in a preliminary way based on synthesized phantom data (Fig 2). The algorithm used is a trust-region nonlinear optimization algorithm we developed. The image size in this simulation is 64x64.

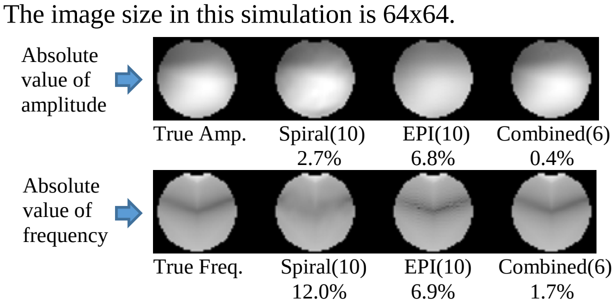


Fig 2. Simulation results. The numbers in brackets represent iteration counts. The percentages represent normalized MSE.

REFERENCES

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